Congestion load balancing game with losses

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***Abstract*—We study the symmetric version of the load balancing game introduced by H. Kameda. We consider a non-splittable atomic game with lossy links. Thus costs are not additive and flow is not conserved (total flow entering a link is greater than the flow leaving it). We show that there is no unique equilibrium in the game. We identify several symmetric equilibria and show how the number of equilibria depends on the problem's parameters. We compute the globally optimal solution and compare its performance to the equilibrium. We finally identify the Kameda paradox which was introduced initially in networks without losses.**

***Index Terms*—Routing games, Loss probabilities, load balancing game, Kameda paradox**

I. Introduction

We study a congestion type load balancing game in which traffic originates from two source nodes and is destined to the center node. Each source node has two possible paths: either a direct path from the source node to the destination node, or a two hop path in which the packet is first relayed to the other source node and then takes the direct link from that node to the destination.There are $2N$ players, N of which arrive at the left source node and N to the right one. The traffic originating from a given player is assumed to form an independent Poisson process with some intensity (which we call the demand). The demand is assumed in this paper to be the same for all players. A player cannot split its traffic. It has to decide whether all its traffic should be transmitted over the direct path (DP) or the indidirect path (IP).

Routing games with general topologies have been intensively studied both in the road traffic community [6], [14], [18] as well as in the community of telecommunications network [13] under additive costs (such as delays or tolls) and conservation constraints (at each node, the sum of incoming flow equals the sum of outgoing flows). In this paper we depart from these assumption by considering loss networks in which losses may occur at all links: there are links with i.i.d. losses (relay links) and collision losses (on direct links between a source and the common destination node).

A. Contribution

Little is known in routing games in the case where some of these assumptions fail to hold. Nonadditive costs in the shortest path problem were studied in [7], [8] (the cost here depends on the path but not on the congestion). Important classes of nonadditive cost problems occurring in telecommunication networks, are those that deal with losses. The two main frameworks to consider losses are in (1) circuit switching networks, in which calls that do not find sufficient resources on each link on a path from the source to the destination are rejected. Routing games with this type of cost were studied in [5] and in [1]; (2) packet switching networks, in which packet losses may occur either due to buffer overflows (these are congestion losses of packets) or random non-congestion losses that are due to the transmission channel (e.g., a radio channel).

In this paper, two levels of system modeling are presented: a flow level in which routing decisions are taken, and a more detailed packet level modeling which determines the losses and thus the interference between flows from different players. The framework we consider is a combination of packeet and circuit switching. We consider a non-splittable framework where each player has the same demand and has to decide on the route that all his packets will follow. The framework we study is thus close to the one of [16], [17]. A general framework is introduced and studied there in which each player has to decide on the path in the network between a source and a destination. The existence of an equilibrium in pure strategies is established there. Yet, since our performance measure is not additive and flow is not conserved, we cannot use the existing result. In particular, we do not know whether the game with losses has a potential.

We study the symmetric load-balancing routing game with the triangular topology. This game was introduced by Kameda, see [10] and references therein. It was further studied in [3], [4] under "classical" assumptions on the model (splitable game, and/or additive costs).

Comparing with the splittable model in which there is a single equilibrium [3], [10], we find in this paper several ordered symmetric pure equilibria; their number depends on the system parameters. We derive the globally optimal policy and show that it is always an equilibrium (but there may ne other non-optimal equilibria). This is in contrast with the splittable game in which there is a single equilibrium which coincides with the globally optimal solution only for large enough exogeneous losses.

II. The model

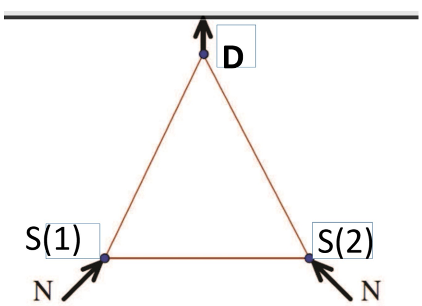

Fig. 1. Physical System

Consider a network with three nodes: $S(1), S(2), D$ depicted in Fig. 1. There are $2N$ players each of which is identified with a flow that is associated with that player. Each player controls the route to be followed by all packets of the flow. There are two sets, N_1 and N_2 each containing N flows originating at each one of the source nodes $S(i), i = 1, 2$. We denote by (n, i) player n among the N_i connected to $S(i)$. All flows have D as a common destination.

Each flow can choose two possible routes (actions): Direct Path (DP) and Indirect one (IP). The DP for a flow arriving at S(i) is link L(i) between S(i) and the destination node D. The indirect path has two hops: a bidirectional relay R from S(i) to S(j) and then the link L(j) from $S(j)$ to D.

Loss probabilitis We consider two types of losses: (1) i.i.d. losses at the relay. A packet originating from node $S(i)$ and f relayed to node $S(j)$ is lost with probability q. (2) collision losses on the links $L(i)$, $i = 1, 2$: whenever an arrival occurs while there is another packet in service then there is a loss. The transmission duration of a packet in link $L(i)$ is exponentially distributed with paramter μ.

The total flow sent to the link $L(i)$ consists of the superpsition of (1) The Poisson flows that arrive at node $S(i)$ and are transmitted over $L(i)$, (2) The Poisson flow originated in node $S(j)$ consisting of the packets that were not lost in the relay to node i.

Consider a matrix valued function u whose 2×N components $\{u(n, i)\}$ take values DP or IP. u is called a pure multi-strategy. Let $\alpha_i(u)$ denote the fraction of players in N_i that choose DP, $i = 1, 2$ under u. Thus, $\alpha_i(u) = \sum_{n=1}^{N} u(n, i)$.

The loss probabilities experienced by a player (n, i) depends on the actions $u(m, i)$ of all other players only through $\alpha_i(u)$. We shall thus often call α "multi-policies" We say that a policy u is symmetric if $\alpha_i(u) = \alpha_j(u) =: \alpha$.

Under a symmetric policy, the total rate of arrivals to $L(i)$ is

$$R(\alpha, i) = N\alpha + N(1-\alpha)(1-q) = N(1-(1-\alpha)q). \tag{1}$$

The loss probability of packets at $L(i)$ is

$$P(\alpha, i) = \frac{R(\alpha)}{R(\alpha) + \mu},$$

Under the symmetric policy u, the probability of a packet loss of a player (n, i) given that it routes its own flow to its DP is given by

$$J_i^n(\alpha) = P(\alpha, i)$$

If it routes the flow through the IP then the loss probability is

$$J_i^n(\alpha) = q + (1-q)P(\alpha, j).$$

III. Global oprimization and equilibria

A. Global optimization

We next show that the symmetric policy that minimizes the loss probabilities is the one for which only DP is used. Consider a symmetric policy α for all players. The loss probability of an arrival is

$$\begin{aligned} P_\alpha &= \frac{R(\alpha)}{R(\alpha)+\mu}\Big[\alpha + (1-\alpha)(1-q)\Big] + q(1-\alpha) \\ &= \frac{R(\alpha)}{R(\alpha)+\mu} + q(1-\alpha)\frac{\mu}{R(\alpha)+\mu} \qquad (2) \\ &= 1 + (q(1-\alpha)-1)\frac{\mu}{R(\alpha)+\mu} \end{aligned}$$

Differentiating w.r.t. α we obtain

$$\begin{aligned} \frac{\partial P_\alpha}{\partial \alpha} &= \mu q \frac{-R(\alpha) - \mu + (1-q(1-\alpha))Nq}{(R(\alpha)+\mu)^2} \\ &= -\frac{\mu^2 q}{((1+(a-1)q)N+\mu)^2} < 0 \end{aligned}$$

This implies that P_α decreases in α and is thus minimized at $\alpha = 1$.

B. Symmetric equilibria

u is a Nash equilibrium if and only if the following two conditions hold: 1. Any player that chose DP does not gain by deviating unilaterally to IP. 2. Any player that chose IP does not gain by deviating unilaterally to DP.

Assume that player (n, i) deviates from DP to IP. This results in a change of $R(i)$ from the expression in (1) to

$$\hat{R}(j) = N\alpha + (1 + N(1-\alpha))(1-q) \qquad (3)$$

and hence in a change in the loss rate of player (n, i) from $P(i)$ to

$$\begin{aligned} \hat{J}_i^n &= (1-q)\frac{\hat{R}(j)}{\hat{R}(j)+\mu} + q \\ &\geq (1-q)\frac{\hat{R}(j)}{\hat{R}(j)+\mu} + q\frac{\hat{R}(j)}{\hat{R}(j)+\mu} \\ &= \frac{\hat{R}(j)}{\hat{R}(j)+\mu} = P(\alpha, i) \end{aligned}$$

Thus no player that uses DP can gain by deviating to an IP. This implies that

Theorem 1: The symmetric multi-policy $\alpha = 1$ is an equilibrium. Hence the price of stability is 1.

Now consider the symmetric policy α and let there be a player that uses an IP and deviates to a DP. Before the deviation his loss rate equals

$$\tilde{P} = q + (1-q)\frac{N\alpha + N(1-\alpha)(1-q)}{N\alpha + N(1-\alpha)(1-q) + \mu}.$$

After the deviation, her loss rate is

$$\overline{P} = \frac{\alpha N + 1 + (1-\alpha)N(1-q)}{\alpha N + 1 + (1-\alpha)N(1-q) + \mu}$$

We conclude that α is a symmetric equilibrium if $\Delta := \overline{P} - \tilde{P} \geq 0$. Solving this gives

$$\alpha \geq Z\Big(\frac{Nq^2 - Nq - \mu q - q + 1}{Nq^2}\Big) =: \alpha^* \qquad (4)$$

where $Z(x)$ is the smallest integer multiple of 1/N larger than or equal to x.

We see in particular that

- all symmetric policies are equilibria for all q sufficiently small,
- For all N sufficiently large, the globally optimal policy is the unique equilibrium.
- For any $\alpha_0 > 0$ there is some N_0 such that there is no symmetric equilibrium $\alpha < \alpha_0$ for all $N > N_0$.
- In particular, the worst (i.e. the smallest) symmetric equilibrium α^* is increasing in N. and converges to 1.

IV. Numerical results

In the Fig. 2 we show Δ as a function of α for $N = 10$, $\mu = 1$ and for four different values of q.

For $q = 0.1$, Δ is negative for all α. Thus, there is no other equilibrium than the trivial one, $\alpha = 1$.

On the other extreme, for $q = 0.003$, all $\alpha > 0$ which are multiples of 1/N are symmetric equilibria as Δ is seen to be positive for all these α.

For $q = 0.07$ we see that Δ is positive for $\alpha \geq \alpha^* = Z(0.568) = 0.6$ and are negative for all other α which are integer multiples of 1/N. Thus 0.6, 0.7, 0.8, 0.9 and 1 are the symmetric equilibria.

For $q = 0.05$, $\Delta > 0$ for $\alpha \geq \alpha^* = Z(0.818) = 0.9$ and are negative for other integer multiples of 1/N. Thus only 0.9 and 1 are symmetric equilibria.

We note directly from (4) that if for some N, q, μ, α is a symmetric equilibrium, then also any other $\alpha' < \alpha$ is an equlibrium.

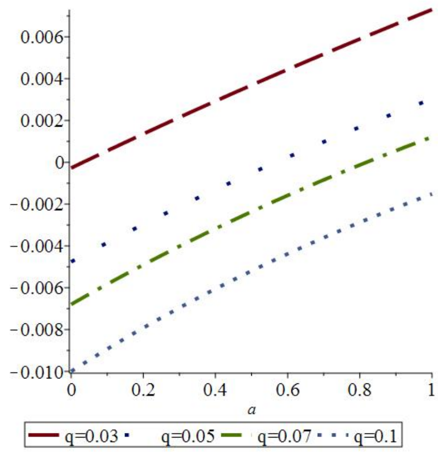

Fig. 2. Δ as a function of α and q

V. Conclusions

Let us compare the results for our non splitable game to those with the same topology and losses as in this paper but in which the traffic of each player is splitable. In the classical splittable framework (of additive cost and conservation of flow) the uniqueness of equilibrium was established in [2]. For the non-classical cost, it was established in [3] (i.e. for the loss probability splittable criteria). In contrast to the above, it follows from our results in this paper that for all q sufficiently small, all N symmetric policies are equilibria.

We have identified the unique globally optimal policy and showed that it is an equilibrium for all systems' paramters. The number of other equilibria decreases with the loss parameter q. This can be interpreted as a Braess type paradox since the performance of the worst equilibrium (the one with highest loss probability of players) improves as the channel quality degrades, i.e. q increases. (Recall that there may be various symmetric equilibria for each set of system parameters.) A similar behavior was already observed in [4] for classical cost model (additive) and for the case of flow conservation.

The original Braess paradox [9] was shown to hold in a framework of a very large number of players (Wardrop equilibrium) and later on it was shown to occur also in the case of any number $N > 1$ of players in [12]. The paradox we studied here known as the Kameda paradox, does not occur in the case of a very large number of players. This was shown for standard delay type cost functions in [10], [11] for triangular network topology.

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